Supersymmetric Localization & Exact Quantum Entropy of Black Holes

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- A.D. João Gomes, Sameer Murthy, *"Localization and Exact Quantum Entorpy of Small Black Holes,"* arXiv:1101.nnnn

Black Hole Entropy

Bekenstein [72]; Hawking[75]

- Entropy of black holes remains one of the most important and precise clues about the microstructure of quantum gravity.
- By now, there is a very good statistical understanding of the entropy of a large class of supersymmetric charged black holes in several compactifications of string theory, in the thermodynamic limit of large horizon area or large charges. Strominger & Vafa [96]
- For a BPS black hole with charge vector Q, the leading Bekenestein-Hawking entropy precisely matches the logarithm of the degeneracy of the corresponding quantum microstates.

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This beautiful approximate agreement raises two important questions:

- What exact formula is this an approximation to?
- Can we systematically compute corrections to both sides of this formula, perturbatively and nonperturbatively in $1/Q$ and may be even exactly for arbitrary finite values of the charges?

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Finite size effects

- We do not know at present which of the phases of string theory might correspond to the real world. For a theory under construction such as string theory, a useful strategy in such a situation is to focus on *universal* properties that must hold in all phases of the theory.

- One universal requirement for a quantum theory of gravity is that in *any* phase of the theory that admits a black hole, it must be possible to interpret the thermodynamic entropy of the black hole as the statistical entropy of an ensemble of quantum states in the Hilbert space of the theory.

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- The leading Bekenstein-Hawking entropy from this point of view is in a sense a bit too universal since it follows from the Einstein-Hilbert action which is the leading low energy effective action in all phases.
- Finite size effects, by contrast, depend on the “phase” or compactification under consideration and the higher derivative terms in the effective action. They thus provide a sensitive probe of short distance degrees of freedom and hence are physicsally very interesting.
- One can hope to learn more about the microscopic degrees of freedom of quantum gravity, effective actions in string theory, nonperturbative functional integral of quantum gravity, exact holography. Analogous to how one might deduce from specific heat of metals whether electrons or phonons are the relevant degrees of freedom.

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Wald entropy and beyond

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- Wald entropy can incorporate the corrections to Bekenstein-Hawking entropy from all higher-derivative *local* terms in the effective action.
- But one should really use the 1PI quantum effective actions which include in general nonanalytic and nonlocal terms.
- These terms are in many cases essential for duality invariance of entropy.

It is thus desirable to have a manifestly duality covariant formalism that generalizes Wald entropy to be able to discuss the finite size effects systematically. Such a generalization has been proposed in the recent work of Sen. We will now review this definition of exact quantum entorpy.

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AdS_2/CFT_1 and exact quantum entropy

- The near horizon geometry of a black hole is $AdS_2 \times S^2$. Following the usual rules of holography leads to a definition of exact quantum entropy Sen [08]
- Consider a black hole with charge vector (q, p). The quantum entropy is defined by a functional integral over all field configurations which asymptote to the AdS_2 Euclidean black hole.
- For a theory with some vector fields A^i and scalar fields ϕ^a, we have the fall-off conditions

$$\begin{aligned} ds_0^2 &= v\left[\left(r^2 + \mathcal{O}(1)\right) d\theta^2 + \frac{dr^2}{r^2 + \mathcal{O}(1)}\right], \\ \phi^a &= u^a + \mathcal{O}(1/r), \qquad A^i = -i\, e^i (r - \mathcal{O}(1)) d\theta, \end{aligned} \tag{1}$$

- Magnetic charges are fluxes on the S^2.

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To start with let us clarify two common misconceptions.

Choice of Ensemble

In two dimensions Coulomb potential grows at the boundary instead of falling. Hence this growing mode must be held fixed and the constant mode can fluctuate.

$$A(r) \sim er + c$$

By Gauss law fixing e means that we are working in the fixed charge sector. Hence, the natural ensemble from the perspective of the AdS_2 boundary conditions is the *microcanonical* ensemble.

Note that this is in contrast to higher dimensional instances of the AdS/CFT correspondence where the constant mode is held fixed.

Index = Degeneracy

If there are least four unbroken supersymmetries then together with the SU(1, 1) symmetry of AdS_2 , closure of algebra implies $SU(1,1|2)$ superalgebra at the horizon. Hence the horizon has an $SU(2)$ symmetry. If J is a generator then microstates associated with the horizon are invariant.

$$Tr[\exp(2\pi i J)] = Tr[1].$$

As a result index equals degeneracy.

Sen [08, 09]
Dabholkar, Gomes, Murthy, Sen [10]

Final definition

The quantum entropy is then defined in terms of the functional integral with an insertion of the Wilson line:

$$W(q,p) = \left\langle \exp\left[-\, i\, q_i \int_0^{2\pi} A^i \, d\theta \right] \right\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} .$$

The constants v, e^i, u^a which set the boundary conditions of the functional integral are determined purely in terms of the charges by the *attractor mechanism*. Hence they must be set to their attractor values v_*, e^i_*, u^a_* . The quantum entropy is thus purely a function of the charges (q, p).

The action in the functional integral suffers from an infrared divergence due to infinite volume of the AdS_2. To obtain a well-defined functional integral one must regulate and renormalize. Holographic renormalization.

Renormalized functional integral

- Put a cutoff at a large $r = r_0$. Let $\mathcal{L}_{bulk}$ is the full local classical Lagrangian density of the theory including all massive fields.
- Since $\mathcal{L}_{bulk}$ is a local functional of the fields, the bulk effective action has the form

$$S_{\mathrm{bulk}} = C_0 r_0 + C_1 + \mathcal{O}(r_0^{-1}) \,, \tag{2}$$

with C_0, C_1 independent of r_0. The linear divergence can then be removed by a boundary counter-term corresponding to a boundary cosmological constant.
- With this prescription, in the semi-classical limit one obtains

$$W(q,p) \sim \exp[S_{wald}(q,p)] \,.$$

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Our main goal is to put these formal definitions to use in concrete examples. Our objectives will be two-fold:

- Compute $W(q, p)$ for arbitrary finite charges by evaluating the functional integral of string field theory on the AdS_2 background.
- Compute $d(q, p)$ from bound state dynamics of branes and check if it equals $W(q, p)$ computed above.

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- Evaluating the formal expression for W(q,p) by doing the string field theory functional integral is of course highly nontrivial.
- It may seem foolishly ambitious to try to evalute the functional integral of full string field theory on the black hole background.
- It turns out that using localization techniques one can go surprisingly far and reduce the functional integral to an ordinary integral.
- With enough supersymmetry, it seems possible to in fact evaluate both $d(q,p)$ and $W(q,p)$ exactly.

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Localization

Consider a supermanifold $\mathcal{M}$ with an integration measure $d\mu$. Let Q be an odd vector field on this manifold satisfying two requirements:

- $Q^2 = H$ for some compact bosonic vector field H,
- The measure is invariant under Q, in other words $div_\mu Q = 0$.

We would like to evaluate an integral of some Q-invariant function h

$$I := \int_{\mathcal{M}} d\mu \, h \, e^{-\mathcal{S}}.$$

To evaluate this integral using localization, one first deforms the integral to

$$I(\lambda) = \int_{\mathcal{M}} d\mu \, h \, e^{-\mathcal{S} - \lambda QV} \ ,$$

where V is a fermionic, H-invariant function.

- The integral $I(\lambda)$ is independent of λ because

$$\frac{d}{d\lambda}\int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}-\lambda QV} = \int_{\mathcal{M}} d\mu\, h\, QV\, e^{-\mathcal{S}-\lambda QV} = 0 \ ,$$

- This implies that one can perform the integral $I(\lambda)$ for any value of λ and in particular for $\lambda \to \infty$.
- Treating $1/\lambda$ as $\hbar$, one can evalute the functional integral semiclassically. Semiclassical approximation is exact.
- The functional integral localizes onto the critical points of the functional $S^Q := QV$ which we refer to as the localizing solutions. This reduces the functional integral over field space to a subspace.

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To apply localization to our problem we will proceed in several steps.

Step I: Integrate out massive string fields

- Integrate out the infinite tower of massive string modes and massive Kaluza-Klein modes to obtain a local Wilsonian effective action for the massless supergravity fields.
- Our problem is reduced to evaluate exactly this functional integral of a finite number of massless fields with AdS2 boundary conditions using the full Wilsonian effective action keeping all higher derivative terms which can include in general not only perturbative corrections in but also worldsheet instanton corrections.
- String theory provides a finite, supersymmetric, and consistent cutoff at the string scale. The functional integral with such a finite cut-off and a Wilsonian effective action will be our starting point.

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Step II: Solve a supergravity problem

- Consider a simpler problem of evaluating $\hat{W}(q,p)$ which is the functional integral of supergravity consisting of only the gravity multiplet and n_v+1 vector multiplets with the AdS_2 boundary conditions.
- This is still a complicated functional integral. We will show that this functional integral localizes onto an ordinary integral over n_v+1 real parameters leading to an enormous simplification.
- If the action consists of only F-terms then the classical part of the integrand is given by the OSV partition function.
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Step III: Back to $W(q,p)$

Use the results in Step II to evaluate $W(q,p)$

- There are nonperturbative contributions from orbifolds of AdS_2 of order s. As a result, $W(q,p)$ has the following form

$$W(q,p) = \sum_s W_s(q,p)\,.$$

- If D-terms and hyper do not contribute for reasons of supersymmetry, $W_1(q,p) = \hat{W}(q,p)$. If some D-terms do contribute, their contribution can be taken into account systematically.
- Evaluation $W_s(q,p)$ for $s \neq 1$ is related to the problem of evaluation of $\hat{W}(q,p)$ in a simple way.

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- Evaluation $W_s(q,p)$ for $s \neq 1$ is related to the problem of evaluation of $\hat{W}(q,p)$ in a simple way.

Step III: Back to $W(q, p)$

Use the results in Step II to evaluate $W(q, p)$

- There are nonperturbative contributions from orbifolds of AdS_2 of order s. As a result, $W(q, p)$ has the following form

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To apply localization to our problem, we need to identify a fermionic symmetry Q. Near horizon geometry has $SU(1,1|2)$ symmetry.

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\begin{aligned}
[L, L_\pm] &= \pm L_\pm \,, & [L_+, L_-] &= -2L \,, \\
[J, J^\pm] &= \pm J^\pm \,, & [J^+, J^-] &= 2J \,, \\
[L, G^{ia}_\pm] &= \pm\frac{1}{2} G^{ia}_\pm \,, & [L_\pm, G^{ia}_\mp] &= -iG^{ia}_\pm \,, \\
[J, G^{i\pm}_r] &= \pm\frac{1}{2} G^{i\pm}_r \,, & [J^\pm, G^{i\mp}_r] &= G^{i\pm}_r \,, \\
\{G^{i\pm}_+, G^{j\pm}_-\} &= \pm 4\epsilon^{ij} J^\pm \,, & \{G^{i+}_\pm, G^{j-}_\pm\} &= \mp 4i\epsilon^{ij} L_\pm \,,
\end{aligned}
$$

$$
\{G^{i+}_\pm, G^{j-}_\mp\} = 4\epsilon^{ij}(L \mp J) \,.
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ds^2 = v\left[d\eta^2 + \sinh^2(\eta)d\theta^2\right] + v\left[d\psi^2 + \sin^2(\psi)d\phi^2\right] \,.
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Choice of the Supercharge

Pick an element

$$Q = G_{+}^{++} + G_{-}^{--}$$

which squares to $H = 4(L - J)$. Here L generates rotation of the Euclidean AdS_2 which is a disk and J generates a rotation of S^2. As a result, H is a compact generator.

This corresponds to choosing the following combination of Killing spinors as susy transformation parameter.

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Given this choice of Q we choose the localizing action functional to be

$$S^Q = QV; \qquad V = (Q\Psi, \Psi)$$

where Ψ denotes schematically *all* fermions of the theory. $Q\Psi$ is thus the supersymmetry transformation using the particular spinor combination above. Pestun[07], Banerjee et. al [09]

Since we are applying localization inside the functional integral, it is necessary to use off-shell formulation supergravity. In general, off-shell supergravity is notoriously complicated, but for vector multiplets an elegant formalism exists using superconformal calculus where one gauges the full superconformal group.

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Supergravity multiplets

- *Weyl multiplet*: The field content is:

$$\mathbf{w} = \left(e_\mu^a, w_\mu^{ab}, \psi_\mu^i, \phi_\mu^i, b_\mu, f_\mu^a, A_\mu, \mathcal{V}_\mu{}^i{}_j, T_{ab}^{ij}, \chi^i, D \right) . \qquad (3)$$

Contains the vielbein, spin connection, auxiliary fields and fermions.

- *Vector multiplet*: The field content is

$$\mathbf{X}^I = \left(X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I \right) \qquad (4)$$

Here X^I is a complex scalar,, A_μ^I a vector field, and Y_{ij}^I are an $SU(2)$ triplet of auxiliary scalars.

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Off-shell supersymmetry transformations

$$\delta\Omega_i = 2\not{D}X\epsilon_i + \frac{1}{2}\varepsilon_{ij}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i \, ,$$

Here i is the $SU(2)$ doublet index and η is the superconformal supersymmetry,

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Importance of being off-shell

- The beauty of off-shell supergravity consists in the fact that the supersymmetry transformations are specified once and for all and do not depend on the choice of the action.
- Much like the coordinate transformation of the specified from general covariance and does not depend on the what action we use.
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Parametrization of the fields

- $$X^I := X^I_* + H^I + iJ^I\,, \qquad \bar{X}^I := \bar{X}^I_* + H^I - iJ^I$$

- Note that Y^I_{ij} are triplets under the $SU(2)$ rotation. It will turn out that for the BPS equations that we solve, they all have to be aligned along the same direction in the $SU(2)$ space. Hence we parametrize them as

$$Y^{I1}_{\ \ 1} = -Y^{I2}_{\ \ 2} = K^I\,; \qquad Y^{I1}_{\ \ 2} = Y^{I2}_{\ \ 1} = 0\,.$$

- We will similarly denote by $f^I_{\mu\nu}$ the electromagnetic field away from the attractor values.

The bosonic part of the localizing action $(Q\Psi, Q\Psi)$ is then given by

$$
\begin{aligned}
& \cosh(\eta)\left[K - 2\operatorname{sech}(\eta)H)\right]^2 \\
+ \quad & 4\cosh(\eta)\left[H_1 + H\tanh(\eta)\right]^2 + 4\cosh(\eta)[H_0^2 + H_2^2 + H_3^2] \\
+ \quad & 2A\left[f_{01}^- - J - \frac{1}{A}(\sin(\psi)J_3 - \sinh(\eta)J_1)\right]^2 \\
+ \quad & 2B\left[f_{01}^+ + J - \frac{1}{B}(\sin(\psi)J_3 + \sinh(\eta)J_1)\right]^2 \\
+ \quad & 2A\left[f_{03}^- + \frac{1}{A}(\sin(\psi)J_1 + \sinh(\eta)J_3)\right]^2 \\
+ \quad & 2B\left[f_{03}^+ + \frac{1}{B}(\sin(\psi)J_1 - \sinh(\eta)J_3)\right]^2
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$$

$$
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&+ \quad 2A\left[f_{02}^{-} + \frac{1}{A}\left(\sin(\psi)J_0 + \sinh(\eta)J_2\right)\right]^2 \\
&+ \quad 2B\left[f_{02}^{+} - \frac{1}{B}\left(\sin(\psi)J_0 + \sinh(\eta)J_2\right)\right]^2 \\
&+ \quad \frac{4\cosh(\eta)}{AB}\left[\sinh(\eta)J_0 - \sin(\psi)J_2\right]^2 \\
&+ \quad \frac{4\cosh(\eta)\sinh^2(\eta)}{AB}\left[J_1^2 + J_3^2\right],
\end{aligned}
$$

where

$$
H_a^I := e_a^\mu \partial_\mu H^I\,, \quad J_a^I := e_a^\mu \partial_\mu J^I\,,
$$

$$
A := \cosh(\eta) + \cos(\psi)\,, \qquad B := \cosh(\eta) - \cos(\psi)\,.
$$

It is understood that all squares are summed over the index I.

- Note that this action is a sum of complete squares. Thus, to obtain the critical points of this localizing action, we set each of the terms in square bracket to zero. This gives a set of first order differential equations.
- It turns out one can solve these equations exactly subject to the boundary conditions of AdS_2 to obtain an explicit analytic form for the localizing instantons.
- This family of instanton solutions is labeled by $n_v + 1$ real parameters C^I, $I = 0, \ldots, n_v$. We have thus solved a major piece of the problem that we set out to solve.

We have explicitly identified the off-shell field configurations onto which the functional integral localizes labeled by finite number of real parameters. We have thus successfully reduced the infinite dimensional functional integral to a finite dimensional ordinary integral.

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Localizing instanton Solution

- For these solutions, the scalar fields X^I in the vector multiplets are no longer fixed at the attractor values X^I_* but have a nontrivial position dependence in the interior of the AdS_2 given by

$$X^I = X^I_* + \frac{C^I}{\cosh(\eta)}\,, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{\cosh(\eta)}$$

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We now need to evaluate the physical action on the localizing instantons after proper renormalization to compute $S_{ren}(C, q, p)$

$$
\begin{aligned}
& (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2} R) + [i \nabla_\mu F_I \nabla^\mu \bar{X}^I \\
+ \; & \frac{1}{4} i F_{IJ} (F^{-I}_{ab} - \frac{1}{4} \bar{X}^I T^{ij}_{ab} \varepsilon_{ij})(F^{-abJ} - \frac{1}{4} \bar{X}^J T^{ij}_{ab} \varepsilon_{ij}) \\
- \; & \frac{1}{8} i F_I (F^{+I}_{ab} - \frac{1}{4} X^I T_{abij} \varepsilon^{ij}) T^{ij}_{ab} \varepsilon_{ij} - \frac{1}{8} i F_{IJ} Y^I_{ij} Y^{Jij} - \frac{i}{32} F \, (T_{abij} \varepsilon^{ij})^2 \\
+ \; & \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik} \varepsilon^{jl} \widehat{B}_{ij} \widehat{B}_{kl} - 2 \widehat{F}^{-}_{ab} \widehat{F}^{-}_{ab}) \\
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Renormalized action

- Substituting our localizing instanton solution in the above action we can extract the finite piece after removing the leading divergent piece linear in r_0 by holographic renormalization.
- After a tedious algebra, one obtains a remarkably simple form for the renormalized action S_{ren} as a function of $\{C^I\}$.

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p) \tag{5}$$

with $\phi^I := e_*^I + 2iC^I$ and $\mathcal{F}$ given by

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Remarks

- Note that $S_{ren}(\phi, q, p)$ equals precisely the *classical* entropy function $\mathcal{E}(e, q, p)$. The physics is however completely different.
- In the entropy function, the scalar fields are set at the their attractor values X_* which is their value at the boundary of AdS_2. In the renormalized action, the scalar fields are set at their values at the center of AdS_2. This is very important.
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- The entropy function is an essentially classical on shell object. Only its critical points and the value of the function at these critical points has physical meaning. This fixes only the first two terms in a Taylor expansion.
- In particular, there are in principle an infinite number of functions with the same critical behavior. It is something of a surprise, after a long calculation, that the renormalized action is precisely equal to the entropy function.
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Localization of the functional integral

- The full functional integral of string field theory on AdS_2 localizes onto the submanifold $\mathcal{M}_Q$ of critical points of the functional S^Q where Q is a specific supersymmetry.
- We have obtained exact analytic expression for a family of nontrivial complex instantons as *exact* solutions to the equations of motion that follow from extremization of S^Q.
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- The Wilson line expectation value in supergravity takes the general form

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Relation to OSV

- As we have seen classical entropy function is an essentially on-shell object. OSV made an inspired guess to give it an off-shell status.
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- What enters the renormalized action is the value of the scalar fields at the center of the AdS_2 *which is allowed to fluctuate*. We are able to access large regions in the field space away from critical points because of localization.
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In the next section, we will see that all of these ingredients play an important role and can in some cases be evaluated explicitly *e. g.* with $\mathcal{N} = 4$ supersymmetry. The determinants in this case can equal unity. In these examples, the known microscopic answers provide a very useful guide.

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Exact microscopic degeneracy of half-BPS black holes

- Consider Heterotic string compactified on T^6. Degeneracy of half-BPS states with charge vector q are given by the Fourier coefficients of the partition funciton

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of 24 left-moving transverse bosons of the heterotic string.
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Rademacher expansion

The degeneracy admits and *exact* expansion

$$d(n) = \sum_{c=1}^{\infty} Kl(n; -1; c) \left(\frac{2\pi}{c}\right)^{14} \tilde{I}_{13}(\frac{4\pi\sqrt{n}}{c})$$

where

$$\tilde{I}_{13}(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{1}{t^{14}} e^{t + \frac{z^2}{4t}} dt,$$

is a modifield Bessel function of index 13, and

$$Kl(n; -1; c) = \sum_{d \in (\mathbb{Z} \not{c} \mathbb{Z})^*} \exp(\frac{2\pi i d n}{c}) \cdot \exp(\frac{-2\pi i}{dc}).$$

is called the “Kloosterman sum”. This sum simplifies for $c = 1$ being equal to 1, but for other values of c it shows a nontrivial dependence on n.

Essentially all features of this formula can be reproduced from the *macroscopic* comutation.

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A. D, João Gomes, Sameer Murthy, *work in progress*

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